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SUM OF SETS IN SEVERAL DIMENSIONS

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Let A, B be finite sets in $\mathbb{R}^d$ with $|A|=m\le|B|=n$, and assume that there is no hyperplane containing both a translation of A and a translation of B. Under this condition it is proved that the number of distinct vectors in the form $\{a+b : a\in A, b\in B\}$ is at least $n+dm-d(d+1)/2$. This generalizes results of Freiman (case $A=B$) and Freiman, Heppes, Uhrin (case $A=-B$). A more complicated estimate is also given which yields the exact bound for all $n>2d$.

1. Introduction

Let A, B be finite sets in $\mathbb{R}^d$. We want to find estimates for the cardinality of their algebraic sum

$$A+B=\{a+b : a\in A, b\in B\}.$$

Write $|A|=m$, $|B|=n$. In general nothing more than the obvious $|A+B|\ge m+n-1$ can be asserted, which holds with equality if both A and B are arithmetical progressions with a common difference. However, better estimates are possible if we assume that our sets are *proper d-dimensional*, by which we mean that they are not contained in a hyperplane of lower dimension. (Here we use the term "hyperplane" to denote a translation of a subspace; this is sometimes called an affine hyperplane or an affine subspace.) More exactly, define the *dimension* $\dim A$ of a set $A\subset\mathbb{R}^d$ as the dimension of the smallest hyperplane containing A. Put

$$F_d(m,n)=\min\{|A+B| : |A|=m, |B|=n, \dim(A+B)=d\}, \tag{1.1}$$

$$F'_d(m,n)=\min\{|A+B| : |A|=m, |B|=n, \dim B=d\}, \tag{1.2}$$

$$F''_d(m,n)=\min\{|A+B| : |A|=m, |B|=n, \dim A=\dim B=d\}. \tag{1.3}$$

F_d is defined for $m+n\ge d+2$, F'_d for $n\ge d+1$ and F''_d for $m\ge d+1, n\ge d+1$. F_d and F''_d are obviously symmetric, while F'_d may not be (and, in fact, I can show that for certain values of m,n it is not), and they are connected by the obvious inequalities

$$F_d(m,n)\le F'_d(m,n)\le F''_d(m,n).$$

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Now define another function G_d as follows:

$$G_d(m,n) = n + \sum_{j=1}^{m-1} \min(d, n-j), \quad n \geq m \geq 1 \tag{1.4}$$

and for $m > n$ extend it symmetrically, putting $G_d(m,n) = G_d(n,m)$. In other words, if $n-m \geq d$, then we have

$$G_d(m,n) = n + d(m-1). \tag{1.5}$$

If $0 \leq t = n-m < d$, then for $n > d$ we have

$$G_d(m,n) = n + d(m-1) - \frac{(d-t)(d-t-1)}{2} = n(d+1) - \frac{d(d+1)}{2} - \frac{t(t+1)}{2}, \tag{1.6}$$

and for $n \leq d$

$$G_d(m,n) = n + \frac{(m-1)(2n-m)}{2}. \tag{1.7}$$

(1.5–7) are better for calculating G_d, but some properties will follow easier from (1.4).

Our main result is the following.

Theorem 1. *For all positive integers m, n and d satisfying $m+n \geq d+2$ we have*

$$F_d(m,n) \geq G_d(m,n). \tag{1.8}$$

Corollary 1.1. *If $A, B \subset \mathbb{R}^d$, $|A| \leq |B|$ and $\dim(A+B) = d$, then we have*

$$|A+B| \geq |B| + d|A| - \frac{d(d+1)}{2}. \tag{1.9}$$

Two important cases have been studied previously. Freiman [1, 2, section 1.14] proved (1.9) in the case $A=B$. An easy example shows that (1.9) gives the exact bound. Freiman, Heppes and Uhrin [3] proved (1.9) for $B=-A$. However, this bound is probably not exact, except the cases $d=1$ and 2. My conjecture is

$$|A-A| \geq \left(2d - 2 + \frac{2}{d}\right) m - C_d$$

for $d \geq 4$ and $|A-A| \geq 4.5m - C$ for $d=3$.

We also show that Theorem 1 gives the correct value for all but a finite number of pairs (m,n) for a fixed d.

Theorem 2. *Assume $1 \leq m \leq n$. We have*

$$F_d(m,n) = F'_d(m,n) = G_d(m,n). \tag{1.10}$$

unless either $n < d+1$ or $m \leq n-m \leq d$ (in this case $n \leq 2d$).

The behaviour of F_d for a few small values, the behaviour of $F'_d(m,n)$ for $m > n$ and F''_d remain to be described.

2. The lower estimate

Here we prove Theorem 1.

Lemma 2.1. *Let $2 \le m \le n$, $m+n \ge d+2$, $d \ge 2$. At least one of the following inequalities holds:*

$$F_d(m,n) \ge d+1+F_d(m-1,n-1), \tag{2.1}$$

$$F_d(m,n) \ge n+F_{d-1}(m-1,n-1), \tag{2.2}$$

$$F_d(m,n) \ge n+F_{d-1}(m-1,n), \tag{2.3}$$

$$F_d(m,n) \ge m+F_{d-1}(m,n-1). \tag{2.4}$$

In the extremal case when $m+n=d+2$, (2.1) and (2.2) are not defined, thus either (2.3) or (2.4) must hold. If $m+n=d+3$, then (2.1) cannot hold.

We use (x,y) to denote the scalar product and $\mathbf{0}=(0,\dots,0)$ for the nullvector.

Proof. Take two sets A,B with $|A|=m$, $|B|=n$, $|A+B|=F_d(m,n)$. Take an $e\in\mathbb{R}^d$ such that all the scalar products (e,x), $x\in A\cup B$ are different. Let

$$\min\{(e,a): a\in A\}=(e,a_0), \quad \min\{(e,b): b\in B\}=(e,b_0).$$

We may assume that $a_0=b_0=\mathbf{0}$ (this can be achieved by a shift). Write

$$A=\{\mathbf{0}\}\cup A', \quad B=\{\mathbf{0}\}\cup B'.$$

We know $(e,x)>0$ for all $x\in A'\cup B'$. This yields that $\mathbf{0}\notin A'+B'$ which we shall need later.

First case. $\dim(A'+B')=d$. (This can happen only if $m+n\ge d+4$.) Write $C=(A'\cup B')\setminus(A'+B')$. Obviously $A+B=(A'+B')\cup C\cup\{\mathbf{0}\}$, a disjoint union, hence

$$|A+B|=|A'+B'|+|C|+1. \tag{2.5}$$

We claim that every element of $A+B$ is a finite sum of certain elements of C. For $\mathbf{0}$ we admit the empty sum. Next take any $s\in A+B$, $s\ne\mathbf{0}$. Consider the representations of s in the form

$$s=x_1+\dots+x_k, \quad x_j\in A+B, \quad x_j\ne 0$$

(there is at least one, the trivial representation $s=s$ with $k=1$). Since $\min\{(e,x): x\in A\cup B, x\ne\mathbf{0}\}$ is a positive number c by the choice of e, the number of summands in such a representation is limited by $(e,s)/c$. Now take a representation with the maximal possible k. If $x_j\in A'+B'$ for some j, say $x_j=u+v$, $u\in A'$, $v\in B'$, then replacing x_j by the two vectors u and v we get a longer representation. This contradiction shows that all $x_j\in C$ in the maximal representation. Since $A+B$ is d dimensional, we must have $|C|\ge d$, thus (2.5) yields (2.1).

Second case. $\dim(A'+B')<d$. This means that there is a nontrivial subspace Q of $\mathbb{R}^d$ such that $A'+B'\subset Q+p$ with some vector p. This implies that for suitable vectors q and r, orthogonal to Q, we have

$$A'\subset Q+q, \qquad B'\subset Q+r.$$

Assume that we have chosen the minimal Q; its dimension may be $d-1$ or $d-2$.

At least one of q,r must be different from $\mathbf{0}$, since otherwise $A+B$ would be contained in Q. We distinguish three subcases.

If $\dim Q=d-2$, then q and r must be linearly independent. We have

$$A+B=B\cup(A'+B), \tag{2.6}$$

a disjoint union, since $B\subset Q\cup(Q+r)$ and $A'+B\subset(Q+p)\cup(Q+q+r)$. Since $A'+B$ must have dimension $d-1$ (otherwise by (2.6), $A+B$ would have a dimension $<d$), this leads to (2.3).

If $\dim Q=d-1$ and $q\neq\mathbf{0}$, then we use the inclusion

$$A+B\supset\{\mathbf{0}\}\cup B'\cup(A'+B'),$$

a disjoint union since $B'\subset Q+r$ and $A'+B'\subset Q+q+r$ while $\mathbf{0}$ belongs to neither. This implies (2.2).

Finally, if $\dim Q=d-1$ and $q=\mathbf{0}$, then $r\neq\mathbf{0}$ and we use the decomposition

$$A+B=A\cup(A+B').$$

Here $A\subset Q$ and $A+B'\subset Q+r$, thus they are disjoint and we obtain (2.4). ∎

Proof of Theorem 1. We establish (1.8) by a double induction on d and $m+n$. For $d=1$ it reduces to the obvious inequality $|A+B|\geq m+n-1$. Assume now that $d\geq 2$ and (1.8) holds for $d-1$, and also that it holds for d for all pairs m',n' such that $m'+n'<m+n$ (such pairs do not exist in the extremal case $m+n=d+2$, so then this assumption is empty). We establish (1.8) for d,m,n.

From the previous Lemma and the induction hypothesis we infer that at least one of the following inequalities holds:

$$F_d(m,n)\geq d+1+G_d(m-1,n-1), \tag{2.7}$$

$$F_d(m,n)\geq n+G_{d-1}(m-1,n-1), \tag{2.8}$$

$$F_d(m,n)\geq n+G_{d-1}(m-1,n), \tag{2.9}$$

$$F_d(m,n)\geq m+G_{d-1}(m,n-1). \tag{2.10}$$

If $m+n=d+2$ or $d+3$, then we can exclude (2.7) or (2.8), but we shall not use this extra information.

We show that all the right sides here are $\geq G_d(m,n)$. We use the definition (1.4). Separating the case $j=1$ and then replacing j by $j+1$ we find

$$\begin{aligned}G_d(m,n)&=n+\min(d,n-1)+\sum_{j=2}^{m-1}\min(d,n-j)\\&=n+\min(d,n-1)+\sum_{j=1}^{m-2}\min(d,(n-1)-j)\\&=G_d(m-1,n-1)+\min(d+1,n)\leq G_d(m-1,n-1)+d+1,\end{aligned}$$

which works for (2.7).

In case (2.8), we write the definitions for $G_d(m,n)$ and $G_{d-1}(m-1,n-1)$ and substract:

$$G_d(m,n) - G_{d-1}(m-1,n-1) =$$
$$1 + \sum_{j=1}^{m-2} \big(\min(d, n-j) - \min(d-1, n-1-j)\big) + \min(d, n-m+1).$$

Here each term in the sum is at most 1, thus the value of the complete expression is

$$\leq 1 + (m-2) + (n-m+1) = n,$$

which shows that the right side of (2.8) is at least $G_d(m,n)$.

Since $G_d(m,n)$ is obviously an increasing function of n, the corresponding inequality for (2.9) follows from that for (2.8).

In (2.10), in the case $m=n$ the definition (1.4) does not work for $G_{d-1}(m,n-1)$ and we must use the symmetry. We have then

$$\begin{aligned} G_d(m,n) - G_{d-1}(m,n-1) &= G_d(n,n) - G_{d-1}(n-1,n) \\ &= \sum_{j=1}^{n-2} \big(\min(d, n-j) - \min(d-1, n-j)\big) + \min(d,1) \\ &\leq n-1 = m. \end{aligned}$$

Finally, in (2.10) for $m<n$ we have

$$G_d(m,n) - G_{d-1}(m,n-1) = 1 + \sum_{j=1}^{m-1} \big(\min(d, n-j) - \min(d-1, n-1-j)\big) \leq m.$$

This completes the inductive step. ∎

3. A construction

We construct sets for which $|A+B|$ is small and prove Theorem 2. Let $e_1, \dots, e_d$ be the unit vectors and assume $1 \leq m \leq n$, $n \geq d+1$. We put

$$B = \{0e_1, 1e_1, \dots, (n-d)e_1\} \cup \{e_2, \dots, e_d\}.$$

B is obviously d dimensional and $|B|=n$.

If $n-m \geq d$, we put

$$A = \{0e_1, 1e_1, \dots, (m-1)e_1\}.$$

This set satisfies $|A|=m$. The set $A+B$ consists of the vectors ie_1, $0 \leq i \leq n+m-d-1$ and the vectors $ie_1 + e_j$, $0 \leq i \leq m-1$, $2 \leq j \leq d$, consequently

$$|A+B| = n + d(m-1) = G_d(m,n).$$

Now consider the case $n-m=t<d$. Write $t=d-k$ and assume $k\le m$. Now A is defined by

$$A=\{0e_1, 1e_1, ..., (m-k)e_1\}\cup\{e_2, ..., e_k\}.$$

This set satisfies $|A|=m$. The set $A+B$ consists of the vectors ie_1, $0\le i\le 2(n-d)$, the vectors ie_1+e_j, $0\le i\le n-d$, $2\le j\le d$, finally e_i+e_j, $2\le i,j\le k$, hence

$$\begin{aligned}|A+B| &= 2(n-d)+1+(d-1)(n-d+1)+\frac{k(k-1)}{2}\\ &= n(d+1)-\frac{d(d+1)}{2}-\frac{t(t+1)}{2}=G_d(m,n).\end{aligned}$$

These constructions cover all pairs m,n except those listed in Theorem 2. ∎

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